Reduction of Multidimensional Non-Linear D'Alembert Equations to Two-Dimensional Equations: Ansatzes, Compatibility of Reduction Conditions, Reduced Equations[1]

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Abstract

We study conditions of reduction of multidimensional wave equations – a system of d'Alembert and Hamilton equations. Necessary conditions for compatibility of such reduction conditions are proved. Possible types of the reduced equations and ansatzes are described. We also provide a brief review of the literature with respect to compatibility of the system of d'Alembert and Hamilton equations and construction of solutions for the nonlinear d'Alembert equation.

1. Introduction. This paper is a continuation of research started jointly with W.I. Fushchych in 1990s [1]. We study reduction of the nonlinear d'Alembert equation

$$\Box u = F(u), \tag{1}$$
$$\Box \equiv \partial_{x_0}^2 - \partial_{x_1}^2 - \cdots - \partial_{x_n}^2, \quad u = u(x_0, x_1, \ldots, x_n)$$

by means of the ansatz with two new independent variables [2, 3]

$$u = \varphi(y, z), \tag{2}$$

where y, z are new variables. Henceforth n is the number of independent space variables in the initial d'Alembert equation.

Wide classes of exact solutions of non-linear equations having respective symmetry properties may be constructed by means of symmetry reduction of these equations to equations with smaller number of independent variables or to ordinary differential equations (as to the relevant algorithms and examples see [4, 5, 6, 7]).

Reduction and search for solutions of equation (1) by means of symmetry reduction or utilisation of ansatzes were considered in particular in the papers by M. Tajiri [8], J. Patera, R. T. Sharp, P. Winternitz and H. Zassenhaus [9], W.I. Fushchych and M.I. Serov [10], W.I. Fushchych, L.F. Barannyk and A.F. Barannyk [11] (symmetry reduction of Poincaré–invariant nonlinear equations to two–dimensional equations was specifically considered in [12]).

[1]Collection of Papers Dedicated to 70th Anniversary of W.I. Fushchych, Kyiv, Institute of Mathematics, 2006, p. 81-95.

In the paper by W.I. Fushchych, A.F. Barannyk and Yu.D. Moskalenko [13] symmetry reduction of the equation (1) with $F = u^k$ to two-dimensional equations was considered, as well as symmetry of the respective reduced equations.

It is obvious that the method of symmetry reduction does not provide exhaustive description of all exact solutions for an equation. For this reason it is interesting to look for and to develop other algorithmic methods for search of solutions. One of such methods is reduction of equations by means of special substitutions – ansatzes.

P. Clarkson and M. Kruskal [14] proposed a so-called "direct method" for search of exact solutions of nonlinear partial differential equations, and demonstrated that this method gives wider classes of solutions than the method of symmetry reduction by subalgebras of the invariance algebra of an equation (see also [15, 16] and the papers cited therein). However, application of this method for majority of equations presents considerable difficulties as it requires investigation of compatibility and solution of cumbersome overdetermined systems of equations – reduction conditions of the initial equation by means of the selected ansatz.

The solutions that may be obtained by the direct method are also related to symmetry properties of the equation – Q-conditional symmetry of this equation [6, 17, 18] (symmetries of such type are also called non-classical or non-Lie symmetries [14, 19, 20]).

R.Z. Zhdanov, I.M. Tsyfra and R.O. Popovych in [21] established equivalence of non-classical (conditional) symmetry and of the direct approach (utilisation of an ansatz) to reduction of partial differential equations. The paper by W.I. Fushchych "Ansatz 95" [23] contains a review of results related to reduction of a number of wave equations. In the paper [24] W.I. Fushchych and A.F. Barannyk proposed an alternative for the method of application of ansatzes for equation (1) with a degree nonlinearity.

In [22] G. Cicogna presented an analysis of application of various types of conditional and non-classical symmetry for search of solutions for non-linear partial differential equations.

In contrast to an algorithmic method of symmetry reduction, the method of direct reduction with utilisation of ansatzes or exhaustive description of conditional symmetries (even Q-conditional symmetries) cannot be regarded as algorithmic to the same extent. Majority of papers on application of the direct method are devoted to evolution equations or other equations that contain variables of the order not higher than one for at least one of the independent variables, with not more than three independent variables. In such cases solution of the reduction conditions is relatively simple.

Reduction conditions are much more difficult for investigation and solution in the case of equations containing second and/or higher derivatives for all independent variables, and for multidimensional equations.

In the present paper we consider general reduction conditions of a multidimensional equation (1) by means of a general ansatz with two new independent variables. We found necessary compatibility conditions for the respective reduction conditions – we developed the conditions found in [1]. We also describe respective possible forms of the reduced equations. Thus we proved that the reduced equations may have only a particular form. However, the results obtained in this paper do not allow (in the general case) to realise full application of the direct method. To do that it is necessary to find a general solution of the reduction conditions.

A similar problem was considered earlier for an ansatz with one independent variable

$$u = \varphi(y), \tag{3}$$

where y is a new independent variable.

Compatibility analysis of the d'Alembert–Hamilton system

$$\Box u = F(u), \quad u_\mu u_\mu = f(u) \tag{4}$$

in the three-dimensional space was done by S.B. Collins in [31].

Solutions of the system (4) were investigated in the papers by H. Bateman [25], V.I. Smirnov and V.L. Sobolev [26], N.P. Yerugin [27] (for more detailed review see [29, 30]).

The compatibility condition of the system (4) for $f(u) = 0$ was found in the paper [32].

Complete investigation of compatibility of overdetermined systems of differential equations with fixed number of independent variables may be done by means of Cartan's algorithm [28], however, it is very difficult for practical application even in the case of three independent variables, and not applicable for arbitrary number of independent variables. For this reason some ad hoc techniques for such cases should be used even for search of necessary compatibility conditions.

It is evident that the d'Alembert–Hamilton system (4) may be reduced by local transformations to the form

$$\Box u = F(u), \quad u_\mu u_\mu = \lambda, \quad \lambda = 0, \pm 1. \tag{5}$$

Necessary compatibility conditions of the system (5) for four independent variables were studied by W.I. Fushchych and R.Z. Zhdanov [33] (see also [30]).

Later W.I. Fushchych, R.Z. Zhdanov and the author found necessary compatibility conditions for the system (5) for arbitrary number of independent variables [34]:

Statement 1. *For the system* (5) *(n is arbitrary) to be compatible it is necessary that the function F have the following form:*

$$F = \frac{\lambda \partial_u \Phi}{\Phi}, \quad \partial_u^{n+1}\Phi = 0.$$

W.I. Fushchych, R.Z. Zhdanov and I.V. Revenko [35, 29, 36] found a general solution of the system (5) for three space variables (that is four independent variables), as well as necessary and sufficient compatibility conditions for this system [35]:

Statement 2. *For the system* (5) $(u = u(x_0, x_1, x_2, x_3))$ *to be compatible it is necessary and sufficient that the function F have the following form:*

$$F = \frac{\lambda}{N(u + C)}, \quad N = 0, 1, 2, 3.$$

Reduction of equation (1) by means of the ansatz (2) was considered by W.I. Fushchych, R.Z. Zhdanov and I.V. Revenko in [36] for a special case (when the second independent variable enters the reduced equation only as a parameter), described all respective ansatzes for the case of four independent variables, and found the respective solutions. Some solutions of such type for arbitrary n were also considered by A.F. Barannyk and I.I. Yuryk in [37].

In [38] R.Z. Zhdanov and O.A. Panchak considered reduction of the nonlinear d'Alembert equation by means of ansatz $u = \phi(\omega_1, \omega_2, \omega_3)$, for the case $\Box\omega_1 = 0$, $\omega_{1\mu}\omega_{1\mu} = 0$ (that is ω_1

entered the reduced equation only as a parameter). The respective compatibility conditions were studied and new (non-Lie) exact solutions were found.

Let us note that this case does not include completely the case considered here of the ansatz with two new independent variables.

The Hamilton equation may also be considered, irrespective of the reduction problem, as an additional condition for the d'Alembert equation that allows extending the symmetry of this equation. The symmetry of the system

$$\Box u = F(u), \quad u_\mu u_\mu = 0$$

was described in [39]. In [34] a conformal symmetry of the system (4) was found that is was a new conditional symmetry for the d'Alembert equation. Conditional symmetries for the respective considered ansatzes were also described in [36, 38].

2. Necessary compatibility conditions of the system of the d'Alembert–Hamilton equations for two functions or for a complex-valued function. A specific investigation of reduction of multidimensional equations to two-dimensional ones is of interest as solutions of two-dimensional partial differential equations, including non-linear ones, may be investigated more fully than solutions of multidimensional equations, though such equations equations may have more interesting properties than ordinary differential equations.

E.g. let $y_\mu y_\mu = -z_\mu z_\mu = 1$, $z_\mu y_\mu = \Box y = \Box z = 0$. Then equation (6) has the form

$$\varphi_{yy} - \varphi_{zz} = F(\varphi).$$

If $F(\varphi) = \sin\varphi$, then the reduced equation possesses solitonic solutions. If $F(\varphi) = \exp\varphi$, it has a general solution. Two-dimensional reduced equations also may have interesting properties with respect to conditional symmetry.

Substitution of ansatz (2) equation (1) leads to the following equation:

$$\begin{gathered}\varphi_{yy} y_\mu y_\mu + 2\varphi_{yz} z_\mu y_\mu + \varphi_{zz} z_\mu z_\mu + \varphi_y \Box y + \varphi_z \Box z = F(\varphi) \\ \left(y_\mu = \frac{\partial y}{\partial x_\mu}, \quad \varphi_y = \frac{\partial \varphi}{\partial y} \right),\end{gathered} \tag{6}$$

whence we get a system of equations:

$$\begin{gathered}y_\mu y_\mu = r(y,z), \quad y_\mu z_\mu = q(y,z), \quad z_\mu z_\mu = s(y,z), \\ \Box y = R(y,z), \quad \Box z = S(y,z).\end{gathered} \tag{7}$$

System (7) is a reduction condition for the multidimensional wave equation (1) to the two-dimensional equation (6) by means of ansatz (2).

The system of equations (7), depending on the sign of the expression $rs - q^2$, may be reduced by local transformations to one of the following types:

1) elliptic case: $rs - q^2 > 0$, $v = v(y,z)$ is a complex–valued function,

$$\begin{gathered}\Box v = V(v, v^*), \quad \Box v^* = V^*(v, v^*), \\ v^*_\mu v_\mu = h(v, v^*), \quad v_\mu v_\mu = 0, \quad v^*_\mu v^*_\mu = 0\end{gathered} \tag{8}$$

(the reduced equation is of the elliptic type);

2) hyperbolic case: $rs - q^2 < 0$, $v = v(y,z)$, $w = w(y,z)$ are real functions,

$$\begin{aligned}
&\Box v = V(v,w), \quad \Box w = W(v,w), \\
&w_\mu w_\mu = h(v,w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0
\end{aligned} \tag{9}$$

(the reduced equation is of the hyperbolic type);

3) parabolic case: $rs - q^2 = 0$, $r^2 + s^2 + q^2 \neq 0$, $v(y,z)$, $w(y,z)$ are real functions,

$$\begin{aligned}
&\Box v = V(v,w), \quad \Box w = W(v,w), \\
&v_\mu w_\mu = 0, \quad v_\mu v_\mu = \lambda \ (\lambda = \pm 1), \quad w_\mu w_\mu = 0
\end{aligned} \tag{10}$$

(if $W \neq 0$, then the reduced equation is of the parabolic type);

4) first-order equations: $(r = s = q = 0)$, $y \to v$, $z \to w$

$$\begin{aligned}
&v_\mu v_\mu = w_\mu w_\mu = v_\mu w_\mu = 0, \\
&\Box v = V(v,w), \quad \Box w = W(v,w).
\end{aligned} \tag{11}$$

Let us formulate necessary compatibility conditions for the systems (8)–(11).

Theorem 1. *System* (8) *is compatible if and only if*

$$V = \frac{h(v,v^*)\partial_{v^*}\Phi}{\Phi}, \quad \partial_{v^*} \equiv \frac{\partial}{\partial v^*},$$

where Φ *is an arbitrary function for which the following condition is satisfied*

$$(h\partial_{v^*})^{n+1}\Phi = 0.$$

The function h may be represented in the form $h = \frac{1}{R_{vv^*}}$, where R is an arbitrary sufficiently smooth function, R_v, R_{v^*} are partial derivatives by the respective variables.

Then the function Φ may be represented in the form $\Phi = \sum\limits_{k=0}^{n+1} f_k(v) R_v^k$, where $f_k(v)$ are arbitrary functions, and

$$V = \frac{\sum\limits_{k=1}^{n+1} k f_k(v) R_v^k}{\sum\limits_{k=0}^{n+1} f_k(v) R_v^k}.$$

The respective reduced equation will have the form

$$h(v,v^*)\left(\phi_{vv^*} + \phi_v \frac{\partial_{v^*}\Phi}{\Phi} + \phi_{v^*}\frac{\partial_v \Phi^*}{\Phi^*}\right) = F(\phi). \tag{12}$$

equation (12) may also be rewritten as an equation with two real independent variables $(v = \omega + \theta,\ v^* = \omega - \theta)$:

$$2\widetilde{h}(\omega,\theta)(\phi_{\omega\omega} + \phi_{\theta\theta}) + \Omega(\omega,\theta)\phi_\omega + \Theta(\omega,\theta)\phi_\theta = F(\phi). \tag{13}$$

We will not adduce here cumbersome expressions for Ω, Θ that may be found from (12).

Theorem 2. *System* (9) *is compatible if and only if*

$$V = \frac{h(v,w)\partial_w \Phi}{\Phi}, \quad W = \frac{h(v,w)\partial_v \Psi}{\Psi},$$

where the functions Φ, Ψ *for which the following conditions are satisfied*

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

The function h may be presented in the form $h = \frac{1}{R_{vw}}$, where R is an arbitrary sufficiently smooth function, R_v, R_w are partial derivatives by the respective variables. Then the functions Φ, Ψ may be represented in the form

$$\Phi = \sum_{k=0}^{n+1} f_k(v) R_v^k, \quad \Psi = \sum_{k=0}^{n+1} g_k(w) R_w^k,$$

where $f_k(v)$, $g_k(w)$ are arbitrary functions,

$$V = \frac{\sum\limits_{k=1}^{n+1} k f_k(v) R_v^k}{\sum\limits_{k=0}^{n+1} f_k(v) R_v^k}, \quad W = \frac{\sum\limits_{k=1}^{n+1} k g_k(w) R_w^k}{\sum\limits_{k=0}^{n+1} g_k(w) R_w^k}.$$

The respective reduced equation will have the form

$$h(v,w)\left(\phi_{vw} + \phi_v \frac{\partial_w \Phi}{\Phi} + \phi_w \frac{\partial_v \Psi}{\Psi}\right) = F(\phi). \tag{14}$$

Theorem 3. *System* (10) *is compatible if and only if*

$$V = \frac{\lambda \partial_v \Phi}{\Phi}, \quad \partial_v^{n+1}\Phi = 0, \quad W \equiv 0.$$

We cannot reduce equation (1) by means of ansatz (2) to a parabolic equation – in this case one of the variables will enter the reduced ordinary differential equation of the first order as a parameter.

Compatibility and solutions of such system for $n = 3$ were considered in [36]; for this case necessary and sufficient compatibility conditions, as well as a general solution, were found.

System (11) is compatible only in the case if $V = W \equiv 0$, that is the reduced equation may be only an algebraic equation $F(u)$=0. Thus we cannot reduce equation (1) by means of ansatz (2) to a first-order equation.

Proof of these theorems is done by means of utilisation of lemmas similar to those adduced in [33, 34], and of the well-known Hamilton–Cayley theorem, in accordance to which a matrix is a root of its characteristic polynomial.

We will present a brief description of proof of Theorem 2 for the hyperbolic case. For other cases the proof is similar.

We will operate with matrices of dimension $(n+1)\times(n+1)$ of the second variable of the functions v and w:

$$\hat{V} = \{v_{\mu\nu}\}, \quad \hat{W} = \{w_{\mu\nu}\}.$$

With respect to operations with these matrices we utilise summation arrangements customary for the Minkowsky space: $v_0 = i\partial_{x_0}$, $v_a = -i\partial_{x_a} (a = 1, \ldots, n)$, $v_\mu v_\mu = v_0^2 - v_1^2 - \cdots - v_n^2$.

Lemma 1. *If the functions v and w are solutions of the system* (9)*, then the following relations are satisfied for them for any k:*

$$\operatorname{tr}\hat{V} = \frac{(-1)^k}{(k-1)!}(h(v,w)\partial_w)^{k+1}V(v,w),$$
$$\operatorname{tr}\hat{W} = \frac{(-1)^k}{(k-1)!}(h(v,w)\partial_v)^{k+1}W(v,w).$$

Lemma 2. *If the functions v and w are solutions of the system* (9)*, then* $\det\hat{V} = 0$, $\det\hat{W} = 0$.

Lemma 3. *Let $M_k(\hat{V})$ be the sum of principal minors of the order k for the matrix $\hat{V}$. If the functions v and w are solutions of the system* (9)*, then the following relations are satisfied for them for any k:*

$$M_k(\hat{V}) = \frac{(h(v,w)\partial_w)^k\Phi}{k!\Phi}, \quad M_k(\hat{W}) = \frac{(h(v,w)\partial_v)^k\Psi}{k!\Psi},$$

where the functions Φ, Ψ satisfy the following conditions

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

These lemmas may be proved with the method of mathematical induction similarly to [34] with utilisation of the Hamilton–Cayley theorem (E is a unit matrix of the dimension $(n+1)\times(n+1)$).

$$\sum_{k=0}^{n-1}(-1)^k M_k\hat{V}^{n-k} + (-1)^n E\det\hat{V} = 0.$$

It is evident that the statement of Theorem 2 is a direct consequence of Lemma 3 for $k = 1$.

Note 1. Equation (8) may be rewritten for a pair of real functions $\omega = \operatorname{Re} v$, $\theta = \operatorname{Im} v$. Though in this case necessary the respective compatibility conditions would have extremely cumbersome form.

Note 2. Transition from (7) to (8)–(11) is convenient only from the point of view of investigation of compatibility. The sign of the expression $rs - q^2$ may change for various y, z, and the transition is being considered only within the region where this sign is constant.

3. Examples of solutions of the system of d'Alembert–Hamilton equations. Let us adduce explicit solutions of systems of the type (7) and the respective reduced equations. Parameters a_μ, b_μ, c_μ, d_μ ($\mu = \overline{0,3}$) satisfy the conditions:

$$-a^2 = b^2 = c^2 = d^2 = -1 \quad (a^2 \equiv a_0^2 - a_1^2 - \cdots - a_3^2),$$
$$ab = ac = ad = bc = bd = cd = 0;$$

y, z are functions of x_0, x_1, x_2, x_3.

1) $y = ax, \quad z = dx, \quad \varphi_{yy} - \varphi_{zz} = F(\varphi);$

2) $y = ax, \quad z = \left((bx)^2 + (cx)^2 + (dx)^2\right)^{1/2},$

$$\varphi_{yy} - \varphi_{zz} - \frac{2}{z}\varphi_z = F(\varphi);$$

In this case the reduced equation is a so-called radial wave equation, the symmetry and solutions of which were investigated in [44, 45].

3) $y = bx + \Phi(ax + dx), \quad z = cx, \quad -\varphi_{zz} - \varphi_{yy} = F(\varphi);$

4) $y = \left((bx)^2 + (cx^2)\right)^{1/2}, \quad z = ax + dx, \quad -\varphi_{yy} - \dfrac{1}{y}\varphi_y = F(\varphi).$

Conditional symmetry and solutions of various non-linear two-dimensional wave equations that may be regarded as reduced equations for equation (1) were considered in [42]-[46]. It is also possible to see from these papers that symmetry of the two-dimensional reduced equations is often wider than symmetry of the initial equation, that is the reduction to two-dimensional equations allows to find new non-Lie solutions.

4. Conclusions. The results of investigation of compatibility and solutions of the systems (8)–(11) may be utilised for investigation and search of solutions also of other Poincaré–invariant wave equations, beside the d'Alembert equation, e.g. Dirac equation, Maxwell equations and equations for the vector potential.

Any multidimensional equation invariant with respect to the Poincaré algebra (such equations for scalar functions were described in the paper by W.I. Fushchych and the author [47]) may be reduced by means of ansatz (2) to a two-dimensional equation on condition if y and z satisfy the reduction conditions (7).

Thus, in the present paper we found

1) necessary compatibility conditions for the system of the d'Alembert–Hamilton equations for two dependent functions, that is reduction conditions of the non-linear multidimensional d'Alembert equation by means of ansatz (2) to a two-dimensional equation; such compatibility conditions for equations of arbitrary dimensions cannot be found by means of the standard procedure;
2) possible types of the two-dimensional reduced equations that may be obtained from equation (1) by means of ansatz (2).

The found reduction conditions and types of ansatzes may be also used for arbitrary Poincaré–invariant multidimensional equation.

Quite often in his papers W.I. Fushchych along with new results and ideas adduced lists of problems that might develop the obtained results. Supporting this tradition I would also like to adduce the list of the following problems that may develop the investigation presented in this paper.

1. Study of Lie and conditional symmetry of the system of the reduction conditions (7) (symmetry of the system of the d'Alembert equations for the complex function was investigated in [49]).
2. Investigation of Lie and conditional symmetry of the reduced equations (12) and (14), and of the possible first-order reduced equations. Finding of exact solutions of the reduced equations.
3. Relation of the equivalence group of the class of the reduced equations with symmetry of the initial equation.
4. Group classification of the reduced equations.

5. Finding of sufficient compatibility conditions and of a general solution of the compatibility conditions (7) for lower dimensions ($n = 2, 3$).
6. Finding and investigation of compatibility conditions and classes of the reduced equations for other types of equations, in particular, for Poincaré–invariant scalar equations.

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